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Connecting weakly nonlinear elasticity theories of isotropic hyperelastic materials

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Abstract
Soft materials exhibit significant nonlinear geometric deformations and stress–strain relationships under external forces. This paper explores weakly nonlinear elasticity theories, including Landau's and Murnaghan's formulations, advancing understanding beyond linear elasticity. We establish connections between these methods and extend strain-energy functions to the third and fourth orders in power of ε, where $\varepsilon = \sqrt{\mathbf{H} \cdot \mathbf{H}}$ and $0 < \varepsilon \leq 1$, and $\mathbf{H}$ is the perturbation to the deformation gradient tensor $\mathbf{F} = \mathbf{I} + \mathbf{H}$. Furthermore, we address simplified strain-energy functions applicable to incompressible materials. Through this work, we contribute to a comprehensive understanding of nonlinear elasticity and its relationship to weakly nonlinear elasticity, facilitating the study of moderate deformations in soft material behavior and its practical applications.

Keywords
Weakly nonlinear elasticity, hyperelasticity, incompressible materials, Landau, Murnaghan, invariants

1. Introduction

Soft materials, such as biological tissues and gels, often undergo significant geometric deformations when subjected to external forces [1]. Unlike hard materials, which typically only experience small deformations, the stress–strain relationship in soft materials is best described using nonlinear elasticity theory due to the large deformations involved [2–5]. Linear elasticity theory is often insufficient to accurately represent the stress–strain relationship in these materials, necessitating the use of nonlinear finite-deformation theory and precise constitutive modeling [6].

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For homogeneous isotropic hyperelastic materials, finite-deformation analysis commonly relies on the strain-energy function represented by the three principal invariants of the strain tensor [6]. The stress–strain relationship can be obtained by solving the partial derivatives of the strain-energy function. Under the assumption of incompressibility, the strain energy can be further simplified as a function of two principal invariants. Analytical solutions for large deformation problems of incompressible materials in simple structures have been found based on finite deformation theory [7]. However, obtaining analytical solutions that account for large deformations becomes difficult for complex problems like nonlinear contact [8] and post-buckling analysis [9]. Linear elastic approximations also fail to meet the precision requirements. Instead, weakly nonlinear theory provides an effective approach in such cases [10].

Weakly nonlinear elasticity theory, particularly the formulation proposed by Landau et al. [11] and Murnaghan [12], represents a significant advance in nonlinear elasticity theory. The use of the strain-energy function in polynomial form allows for the precise determination of elastic moduli through curve fitting of experimental data using standard linear regression techniques [10, 13].

Landau's [11] approach, based on the Landau invariants of the Cauchy–Green strain tensor, with terms up to the third and/or higher orders of the strain-energy functions, includes both material and geometric nonlinearity [10]. It finds applications in material science, geophysics, acoustics, and other fields, accurately predicting the mechanical response of materials under realistic loading conditions [14–16].

Murnaghan's [12] framework, which expresses the strain-energy function as a triply-infinite power series in the principal invariants of the Cauchy–Green strain tensor, is also widely used. This approach has shown the ability to solve simple problems involving compressible materials and specific cross-sectional shapes of prisms under incompressible conditions [7]. It approximates the strain-energy function to any desired order in the power-series expansion, utilizing the symmetric functions of principal invariants. In addition, under the assumption of small deformations, the higher-order Murnaghan model extends the classical linear elastic theory into the weakly nonlinear region, providing a robust method. By considering the superposition of displacements of higher orders and substituting them into the motion equations and boundary conditions according to the corresponding orders, the Murnaghan model [12] demonstrates the linearization of the nonlinear problem by neglecting higher-order terms and simplifying solutions to quasi-nonlinear problems [17, 18].

This article aims to organize the definitions of different strains, invariants, and strain-energy functions in these two different weakly nonlinear elastic theories and the various forms of strain-energy functions and material parameters in non-linear elastic theory to establish their relationships. We expand the strain-energy functions in the weakly nonlinear theory up to the third and fourth orders, corresponding to the second-order and third-order elasticity theories. In addition, we consider the simplified strain-energy functions and stress–strain relationships of materials under incompressible conditions, as many soft materials can be assumed to be incompressible.

2. Connections among different strain invariants

In the context of finite deformation, let us consider an elastic body undergoing a finite displacement field u from the reference configuration to the current configuration. The deformation gradient tensor $\mathbf{F}$ is defined by $\mathbf{F} = \mathbf{I} + \mathbf{H}$, where $\mathbf{H}$ is the displacement gradient tensor, defined as $\mathbf{H} = \text{Grad}\,\mathbf{u}$. For weakly nonlinear theory, we shall assume that $\mathbf{H}$ is the perturbation to the deformation gradient tensor and is small, i.e., $\varepsilon = \sqrt{\mathbf{H} \cdot \mathbf{H}}$ and $0 < \varepsilon \leq 1$. Henceforth, the left and right Cauchy–Green strain tensors are

$$\begin{aligned} \mathbf{b} &= \mathbf{F}\mathbf{F}^{\mathrm{T}} = \mathbf{I} + \mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}\mathbf{H}^{\mathrm{T}} = \mathbf{I} + \mathbf{e} + \boldsymbol{\alpha}, \\ \mathbf{C} &= \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{I} + \mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H} = \mathbf{I} + \mathbf{e} + \boldsymbol{\gamma}, \end{aligned} \tag{1}$$

where we separate $\mathbf{e} = \mathbf{H} + \mathbf{H}^{\mathrm{T}}$, $\boldsymbol{\alpha} = \mathbf{H}\mathbf{H}^{\mathrm{T}}$, and $\boldsymbol{\gamma} = \mathbf{H}^{\mathrm{T}}\mathbf{H}$, so that $\mathbf{e}$ is the first-order term, i.e., $\mathbf{e} = O(\varepsilon)$ and $\boldsymbol{\alpha} = \boldsymbol{\gamma} = O(\varepsilon^2)$ are of second-order terms. The Green–Lagrange strain tensor $\mathbf{E}$ is then given by

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \frac{1}{2}\left(\mathbf{e} + \boldsymbol{\gamma}\right). \tag{2}$$

In linear elasticity, providing that the displacement gradient tensor $\mathbf{H}$ is small, and ignoring the second-order terms of $\mathbf{H}$, we obtain the infinitesimal strain tensor

$$\mathbf{E}^{*} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{T}\right) = \frac{1}{2}\mathbf{e}. \tag{3}$$

The strain-energy function of an ideal isotropic elastic material, capable of undergoing finite deformation, can generally be expressed in terms of three independent scalar invariants of the strain tensor. For a general second-order tensor $\mathbf{M}$, the principal invariants are defined by the equation

$$\mathbf{M}^3 - I_M\mathbf{M}^2 + II_M\mathbf{M} - III_M\mathbf{I} = \mathbf{0}, \tag{4}$$

where the invariants of the tensor $\mathbf{M}$ are

$$I_M = \mathrm{tr}(\mathbf{M}), \quad II_M = \frac{1}{2}\left(\mathrm{tr}(\mathbf{M})^2 - \mathrm{tr}\left(\mathbf{M}^2\right)\right), \quad III_M = \det\mathbf{M}. \tag{5}$$

On the other hand, the scalar invariants of the tensor $\mathbf{M}$ can be defined in Landau's form by

$$\bar{I}_1 = \mathrm{tr}(\mathbf{M}), \quad \bar{I}_2 = \mathrm{tr}\left(\mathbf{M}^2\right), \quad \bar{I}_3 = \mathrm{tr}\left(\mathbf{M}^3\right), \tag{6}$$

where $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$ are, respectively, the first-, second-, and third-order terms of $\mathbf{M}$.

Some commonly used invariants and the relationships between them are

$$\begin{aligned}
&I_C = \mathrm{tr}(\mathbf{C}), \quad II_C = \frac{1}{2}\left(\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right), \quad III_C = \det\mathbf{C},\\
&\bar{I}_1 = \mathrm{tr}(\mathbf{E}), \quad \bar{I}_2 = \mathrm{tr}\left(\mathbf{E}^2\right), \quad \bar{I}_3 = \mathrm{tr}\left(\mathbf{E}^3\right),\\
&J_1 = \mathrm{tr}(2\mathbf{E}), \quad J_2 = \frac{1}{2}\left(\mathrm{tr}(2\mathbf{E})^2 - \mathrm{tr}\left(4\mathbf{E}^2\right)\right), \quad J_3 = \det(2\mathbf{E}),\\
&\tilde{J}_1 = \mathrm{tr}(\mathbf{E}), \quad \tilde{J}_2 = \frac{1}{2}\left(\mathrm{tr}(\mathbf{E})^2 - \mathrm{tr}\left(\mathbf{E}^2\right)\right), \quad \tilde{J}_3 = \det\mathbf{E}.
\end{aligned} \tag{7}$$

Among them, I_C, II_C, and III_C are principal invariants of the right Cauchy–Green strain tensor $\mathbf{C}$, which are often used for constructing the strain-energy function for hyperelastic material in fully nonlinear elasticity. When the material is incompressible ($III_C = 1$), the strain-energy function can be simplified to depend solely on I_C and II_C. In addition, the Landau invariants $\bar{I}_1, \bar{I}_2$, and $\bar{I}_3$ represent the first-, second-, and third-order terms of the Green–Lagrange strain tensor $\mathbf{E}$. Similarly, the principal invariants of the Green–Lagrange strain tensor $\mathbf{E}$, namely $\tilde{J}_1, \tilde{J}_2$, and $\tilde{J}_3$, also represent its first-, second-, and third-order terms. These are known as Murnaghan invariants. However, it is more convenient to use another set of Murnaghan invariants, namely J_1, J_2, and J_3, which are the principal invariants of $2\mathbf{E}$ to simplify the derivation of the stress–strain relationships.

Given the often utilization of both fully nonlinear and weakly nonlinear elasticity theories in the derivation of nonlinear deformation problems, it becomes imperative to comprehensively outline the interconnections between the invariants and the strain-energy functions they engender.

2.1. Connections between I_C, II_C, III_C and $\bar{I}_1$, $\bar{I}_2$, $\bar{I}_3$

In this subsection, we shall demonstrate the connections between the principal invariants, I_C, II_C, and III_C, of the right Cauchy–Green strain tensor $\mathbf{C}$ and the Landau invariants, $\bar{I}_1, \bar{I}_2, \bar{I}_3$, of the Green–Lagrange strain tensor $\mathbf{E}$. First, recalling the equations (2) and (7), we have

$$\begin{aligned}
\mathrm{tr}\mathbf{C} &= \mathrm{tr}(\mathbf{I} + 2\mathbf{E}) = 2\bar{I}_1 + 3,\\
\mathrm{tr}\left(\mathbf{C}^2\right) &= \mathrm{tr}\left(4\mathbf{E}^2 + 4\mathbf{E} + \mathbf{I}\right) = 4\bar{I}_2 + 4\bar{I}_1 + 3,\\
\mathrm{tr}\left(\mathbf{C}^3\right) &= \mathrm{tr}\left(8\mathbf{E}^3 + 12\mathbf{E}^2 + 6\mathbf{E} + \mathbf{I}\right) = 8\bar{I}_3 + 12\bar{I}_2 + 6\bar{I}_1 + 3.
\end{aligned} \tag{8}$$

Then, the first and second principal invariants of the right Cauchy–Green strain tensor $\mathbf{C}$ can be expressed as

$$\begin{aligned}
I_C &= \mathrm{tr}\mathbf{C} = 2\bar{I}_1 + 3,\\
II_C &= \frac{1}{2}\left(\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}\left(\mathbf{C}^2\right)\right) = 3 + 4\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2.
\end{aligned} \tag{9}$$

Next, by tracing equation (4), we obtain

$$\mathrm{tr}\left(\mathbf{C}^3\right) - I_C\mathrm{tr}\left(\mathbf{C}^2\right) + II_C\mathrm{tr}\mathbf{C} - 3III_C = 0. \tag{10}$$

Using this equation, the third principal invariant of the right Cauchy–Green strain tensor **C** can be expressed as

$$\begin{aligned} III_C = \det\mathbf{C} &= \frac{1}{3}\left(\mathrm{tr}\left(\mathbf{C}^3\right) - I_C \mathrm{tr}\left(\mathbf{C}^2\right) + II_C \mathrm{tr}\mathbf{C}\right) \\ &= 1 + 2\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2 + \frac{4}{3}\bar{I}_1^3 - 4\bar{I}_1\bar{I}_2 + \frac{8}{3}\bar{I}_3. \end{aligned} \tag{11}$$

Inversely, from equations (9) and (11), we can express the Landau invariants $\bar{I}_1$, $\bar{I}_2$, and $\bar{I}_3$ in terms of the principal invariants I_C, II_C, and III_C as

$$\begin{aligned} &\bar{I}_1 = \frac{1}{2}(-3 + I_C),\ \bar{I}_2 = \frac{1}{4}\left(3 - 2I_C + I_C^2 - 2II_C\right), \\ &\bar{I}_3 = \frac{1}{8}\left(24 - 24I_C + 12I_C^2 - 2I_C^3 - 12II_C + 3I_C II_C + 3III_C\right). \end{aligned} \tag{12}$$

2.2. Connections between $I_C, II_C, III_C, J_1, J_2, J_3$, and $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$

As the two sets of definitions of the Murnaghan invariants J_1, J_2, J_3, and $\tilde{J}_1, \tilde{J}_2, \tilde{J}_3$ in equation (7) are commonly used, we shall first show connections between them and then demonstrate relationships with the principal invariants of the right Cauchy–Green strain tensor **C**.

From equation (7), we obtain

$$\begin{aligned} J_1 &= \mathrm{tr}(2\mathbf{E}) = 2\mathrm{tr}\mathbf{E} = 2\tilde{J}_1, \\ J_2 &= \frac{1}{2}\left(\mathrm{tr}(2\mathbf{E})^2 - \mathrm{tr}\left(4\mathbf{E}^2\right)\right) = \frac{1}{2}\left(4\tilde{J}_1^2 - 4\mathrm{tr}\left(\mathbf{E}^2\right)\right) = 4\tilde{J}_2, \\ J_3 &= \det(2\mathbf{E}) = 8\det\mathbf{E} = 8\tilde{J}_3. \end{aligned} \tag{13}$$

Furthermore, utilizing equations (2) and (7), we can deduce the following relationships between the Murnaghan invariants I_C, II_C, III_C and J_1, J_2, J_3:

$$\begin{aligned} I_C &= \mathrm{tr}(\mathbf{I} + 2\mathbf{E}) = 3 + \mathrm{tr}(2\mathbf{E}) = 3 + J_1, \\ II_C &= \frac{1}{2}\left((J_1 + 3)^2 - \mathrm{tr}\left(4\mathbf{E}^2 + 4\mathbf{E} + \mathbf{I}\right)\right) = 3 + 2J_1 + J_2, \\ III_C &= \det(\mathbf{I} + 2\mathbf{E}) = 1 + J_1 + J_2 + J_3. \end{aligned} \tag{14}$$

Inversely, we have

$$J_1 = I_C - 3,\ J_2 = 3 - 2I_C + II_C,\ J_3 = I_C - II_C + III_C - 1. \tag{15}$$

Moreover, considering the connections between J_i and $\tilde{J}_i (i = 1, 2, 3)$ in equation (13), we have

$$I_C = 3 + 2\tilde{J}_1,\ II_C = 3 + 4\tilde{J}_1 + 4\tilde{J}_2,\ III_C = 1 + 2\tilde{J}_1 + 4\tilde{J}_2 + 8\tilde{J}_3, \tag{16}$$

and

$$\tilde{J}_1 = \frac{1}{2}(I_C - 3),\ \tilde{J}_2 = \frac{1}{4}(3 - 2I_C + II_C),\ \tilde{J}_3 = \frac{1}{8}(I_C - II_C + III_C - 1). \tag{17}$$

2.3. Connections among $\bar{I}_1$, $\bar{I}_2, \bar{I}_3, J_1$, J_2, J_3, and $\tilde{J}_1$, $\tilde{J}_2, \tilde{J}_3$

This subsection explores the connections between the Landau invariants $\bar{I}_1$, $\bar{I}_2, \bar{I}_3$ and the Murnaghan invariants J_1, J_2, J_3. Utilizing equations (9), (11), and (14), the following relationships can be established:

$$\begin{aligned} 2\bar{I}_1 + 3 &= 3 + J_1, \\ 3 + 4\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2 &= 3 + 2J_1 + J_2, \\ 1 + 2\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2 + \frac{4}{3}\bar{I}_1^3 - 4\bar{I}_1\bar{I}_2 + \frac{8}{3}\bar{I}_3 &= 1 + J_1 + J_2 + J_3. \end{aligned} \tag{18}$$

Table 1. Transformations of some commonly used scalar invariants of strain tensors.

	The principal invariants of $\mathbf{C}$	Landau invariants of $\mathbf{E}$	Murnaghan invariants of $2\mathbf{E}$
The principal invariants of $\mathbf{C}$ $I_C = \mathrm{tr}(\mathbf{C})$ $II_C = \frac{1}{2}(\mathrm{tr}(\mathbf{C})^2 - \mathrm{tr}(\mathbf{C}^2))$ $III_C = \det\mathbf{C}$	–	$I_C = 3 + 2\bar{I}_1$ $II_C = 3 + 4\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2$ $III_C = 1 + 2\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2 + \frac{4}{3}\bar{I}_1^3 - 4\bar{I}_2\bar{I}_1 + \frac{8}{3}\bar{I}_3$	$I_C = 3 + J_1$ $II_C = 3 + 2J_1 + J_2$ $III_C = 1 + J_1 + J_2 + J_3$
Landau invariants of $\mathbf{E}$ $\bar{I}_1 = \mathrm{tr}(\mathbf{E})$ $\bar{I}_2 = \mathrm{tr}(\mathbf{E}^2)$ $\bar{I}_3 = \mathrm{tr}(\mathbf{E}^3)$	$\bar{I}_1 = \frac{1}{2}(-3 + I_C)$ $\bar{I}_2 = \frac{1}{4}(3 - 2I_C + I_C^2 - 2II_C)$ $\bar{I}_3 = \frac{1}{8}(24 - 24I_C + 12I_C^2 - 2I_C^3 - 12II_C + 3I_C II_C + 3III_C)$	–	$\bar{I}_1 = \frac{1}{2}J_1$ $\bar{I}_2 = \frac{1}{4}(J_1^2 - 2J_2)$ $\bar{I}_3 = \frac{1}{8}(J_1^3 - 3J_1J_2 + 3J_3)$
Murnaghan invariants of $2\mathbf{E}$ $J_1 = \mathrm{tr}(2\mathbf{E})$ $J_2 = \frac{1}{2}(\mathrm{tr}(2\mathbf{E})^2 - \mathrm{tr}(4\mathbf{E}^2))$ $J_3 = \det(2\mathbf{E})$	$J_1 = I_C - 3$ $J_2 = 3 - 2I_C + II_C$ $J_3 = I_C - II_C + III_C - 1$	$J_1 = 2\bar{I}_1$ $J_2 = 2\left(\bar{I}_1^2 - \bar{I}_2\right)$ $J_3 = \frac{4}{3}\left(\bar{I}_1^3 - 3\bar{I}_1\bar{I}_2 + 2\bar{I}_3\right)$	–

Solving these equations yields

$$\bar{I}_1 = \frac{1}{2}J_1,\ \bar{I}_2 = \frac{1}{4}\left(J_1^2 - 2J_2\right),\ \bar{I}_3 = \frac{1}{8}\left(J_1^3 - 3J_1J_2 + 3J_3\right), \tag{19}$$

and

$$J_1 = 2\bar{I}_1,\ J_2 = 2\left(\bar{I}_1^2 - \bar{I}_2\right),\ J_3 = \frac{4}{3}\left(\bar{I}_1^3 - 3\bar{I}_1\bar{I}_2 + 2\bar{I}_3\right). \tag{20}$$

Analogously, according to equation (13), we can obtain the connections between the Landau invariants $\bar{I}_i (i = 1, 2, 3)$ and Murnaghan invariants $\tilde{J}_i (i = 1, 2, 3)$ as

$$\bar{I}_1 = \tilde{J}_1,\ \bar{I}_2 = \tilde{J}_1^2 - 2\tilde{J}_2,\ \bar{I}_3 = \tilde{J}_1^3 - 3\tilde{J}_1\tilde{J}_2 + 3\tilde{J}_3. \tag{21}$$

and

$$\tilde{J}_1 = \bar{I}_1,\ \tilde{J}_2 = \frac{1}{2}\left(\bar{I}_1^2 - \bar{I}_2\right),\ \tilde{J}_3 = \frac{1}{6}\left(\bar{I}_1^3 + 3\bar{I}_1\bar{I}_2 + 2\bar{I}_3\right). \tag{22}$$

Finally, the summary of transformations for the principal invariants, I_C, II_C, and III_C, of the right Cauchy–Green strain tensor $\mathbf{C}$, the Landau invariants $\bar{I}_1, \bar{I}_2, \bar{I}_3$ of the Green–Lagrange strain tensor $\mathbf{E}$, and the Murnaghan invariants J_1, J_2, J_3 of the strain tensor $2\mathbf{E}$ can be found in Table 1.

3. Weakly nonlinear elasticity for isotropic compressible materials

In fully nonlinear elasticity, the strain-energy function of the isotropic compressible material W is commonly expressed in terms of the three principal invariants (I_C, II_C, and III_C) of the right Cauchy–Green strain tensor $\mathbf{C}$. The Cauchy stress tensor $\mathbf{t}$ can be represented as

$$\mathbf{t} = \frac{2}{\mathcal{J}}\left(\mathbf{b}\frac{\partial W}{\partial I_C} - III_C\mathbf{b}^{-1}\frac{\partial W}{\partial II_C} + \left(III_C\frac{\partial W}{\partial III_C} + II_C\frac{\partial W}{\partial II_C}\right)\mathbf{I}\right), \tag{23}$$

where $\mathbf{b}$ is the left Cauchy–Green tensor. $\mathcal{J} = \det\mathbf{F}$ and $\mathcal{J} = 1$ for incompressible materials. Here, $I_C - 3$, $II_C - 3$, and $III_C - 1$ are all of $O(\varepsilon)$. For weakly nonlinear elasticity, instead of using the principal invariants of $\mathbf{C}$ to construct the strain-energy function and stress tensor, it is more convenient to employ the Landau and Murnaghan invariants for constructing the strain-energy function up to a certain order of expansion.

3.1. Second-order elasticity

3.1.1. Strain-energy functions. In second-order elasticity, we require the terms of $O(\varepsilon^2)$ in the stress and strain tensors. Consequently, the strain-energy function needs to include the third-order terms of $O(\varepsilon^3)$. In terms of Murnaghan invariants, the strain-energy function to third-order terms is

$$W_M = a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3 + O(\varepsilon^4) \tag{24}$$

where $a_1, ..., a_5$ are $O(\varepsilon)$ material constants. $W_{2M} = a_1 J_2 + a_2 J_1^2 = O(\varepsilon^2)$ and $W_{3M} = a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3 = O(\varepsilon^3)$ are the second- and the third-order terms, respectively. Alternatively, third-order energy function can be expressed in terms of Landau invariants as:

$$W_L = \frac{\lambda}{2}\bar{I}_1^2 + \mu \bar{I}_2 + \frac{\bar{A}}{3}\bar{I}_3 + \bar{B}\bar{I}_1\bar{I}_2 + \frac{\bar{C}}{3}\bar{I}_1^3 + O(\varepsilon^4), \tag{25}$$

where μ and λ are the linear Lamé coefficients, and $\bar{A}$, $\bar{B}$, and $\bar{C}$ are $O(\varepsilon)$ elastic material constants. $W_{2L} = \frac{\lambda}{2}\bar{I}_1^2 + \mu\bar{I}_2$ is the second-order term and $W_{3L} = \frac{\bar{A}}{3}\bar{I}_3 + \bar{B}\bar{I}_1\bar{I}_2 + \frac{\bar{C}}{3}\bar{I}_1^3$ is the third-order term.

Referring to the connections between Murnaghan invariants and Landau invariants in equation (20), the third-order strain-energy function in Murnaghan expansion can be rewritten as

$$W_M = (2a_1 + 4a_2)\bar{I}_1^2 - 2a_1\bar{I}_2 + \frac{8a_5}{3}\bar{I}_3 - 4(a_3 + a_5)\bar{I}_1\bar{I}_2 + \left(4a_3 + 8a_4 + \frac{4a_5}{3}\right)\bar{I}_1^3 + O(\varepsilon^4). \tag{26}$$

Thus, we can obtain the following relationships among the material constants:

$$\begin{aligned} &\lambda = 4a_1 + 8a_2,\ \mu = -2a_1, \\ &\bar{A} = 8a_5,\ \bar{B} = -4(a_3 + a_5),\ \bar{C} = (12a_3 + 24a_4 + 4a_5). \end{aligned} \tag{27}$$

Similarly, referring to the connections between Landau invariants and Murnaghan invariants in equation (19), we can rewrite the third-order strain-energy function in Landau expansion as:

$$W_L = -\frac{\mu}{2}J_2 + \left(\frac{\lambda}{8} + \frac{\mu}{4}\right)J_1^2 - \left(\frac{\bar{A}}{8} + \frac{\bar{B}}{4}\right)J_1J_2 + \left(\frac{\bar{A}}{24} + \frac{\bar{B}}{8} + \frac{\bar{C}}{24}\right)J_1^3 + \frac{\bar{A}}{8}J_3 + O(\varepsilon^4). \tag{28}$$

This leads to

$$\begin{aligned} &a_1 = -\frac{\mu}{2},\ a_2 = \frac{\lambda}{8} + \frac{\mu}{4}, \\ &a_3 = -\left(\frac{\bar{A}}{8} + \frac{\bar{B}}{4}\right),\ a_4 = \frac{\bar{A}}{24} + \frac{\bar{B}}{8} + \frac{\bar{C}}{24},\ a_5 = \frac{\bar{A}}{8}. \end{aligned} \tag{29}$$

3.1.2. Stress–strain relationships. Second-order elasticity requires the stress and strain tensors to be expanded to $O(\varepsilon^2)$. According to equation (23), the Cauchy stress tensor involves the three principal invariants of the right Cauchy–Green strain tensor (I_C, II_C, and III_C), as well as the partial derivatives of the strain-energy function W with respect to them, the left Cauchy–Green strain tensor $\mathbf{b}$, and its inverse tensor $\mathbf{b}^{-1}$. For isotropic elastic solids, the principal invariants of the left Cauchy–Green strain tensor $\mathbf{b}$ are equal to those of the right Cauchy–Green strain tensor $\mathbf{C}$, i.e., $I_b = I_C$, $II_b = II_C$, and $III_b = III_C$.
From equation (1) and defining

$$e = \mathrm{tr}\mathbf{e},\ \alpha = \mathrm{tr}\boldsymbol{\alpha},\ \mathbf{K} = (\det \mathbf{e})\mathbf{e}^{-1}, \tag{30}$$

the first principal invariant I_b can be expressed as

$$I_b = \mathrm{tr}\mathbf{b} = 3 + e + \alpha, \tag{31}$$

and

$$\mathbf{b}^2 = (\mathbf{I} + \mathbf{e} + \boldsymbol{\alpha})^2 = \mathbf{I} + 2\mathbf{e} + 2\boldsymbol{\alpha} + \mathbf{e}^2 + O\left(\varepsilon^3\right). \tag{32}$$

By tracing this equation and using equation (30), we obtain

$$\operatorname{tr}\left(\mathbf{b}^2\right) = 3 + 2e + \alpha + e^2 - 2K + O\left(\varepsilon^3\right), \tag{33}$$

where $K = \operatorname{tr}\mathbf{K} = \frac{1}{2}\left((\operatorname{tr}\mathbf{e})^2 - \operatorname{tr}\left(\mathbf{e}^2\right)\right)$. Thus, the second principal invariant II_b is given as

$$II_b = \frac{1}{2}\left((\operatorname{tr}\mathbf{b})^2 - \operatorname{tr}\left(\mathbf{b}^2\right)\right) = 3 + 2e + 2\alpha + K + O(\varepsilon^3), \tag{34}$$

and the third principal invariant III_b is

$$III_b = \det\mathbf{b} = \det(\mathbf{I} + \mathbf{e} + \boldsymbol{\alpha}) = 1 + j_1 + j_2 + j_3, \tag{35}$$

where $j_1 = \operatorname{tr}(\mathbf{e} + \boldsymbol{\alpha})$, $j_2 = \frac{1}{2}\left(j_1^2 - \operatorname{tr}\left((\mathbf{e} + \boldsymbol{\alpha})^2\right)\right)$, $j_3 = \det(\mathbf{e} + \boldsymbol{\alpha})$. Referring to equation (30) and expanding to second-order terms, we have

$$j_1 = e + \alpha + O\left(\varepsilon^3\right), \; j_2 = K + O\left(\varepsilon^3\right), \; j_3 = 0 + O\left(\varepsilon^3\right). \tag{36}$$

Thus, the third principal invariant III_b can be expanded as

$$III_b = 1 + e + \alpha + K + O\left(\varepsilon^3\right) \tag{37}$$

In addition, using the Cayley–Hamilton theorem and equations (31), (32), and (34), we have

$$\begin{aligned} III_C\mathbf{b}^{-1} &= III_b\mathbf{b}^{-1} = \mathbf{b}^2 - I_b\mathbf{b} + II_b\mathbf{I} \\ &= (\mathbf{e} - e - 1)\mathbf{e} - \boldsymbol{\alpha} + (1 + e + \alpha + K)\mathbf{I} + O\left(\varepsilon^3\right). \end{aligned} \tag{38}$$

Analogously, from equation (30), we have

$$\mathbf{K} = (\det\mathbf{e})\mathbf{e}^{-1} = \mathbf{e}^2 - I_e\mathbf{e} + II_e\mathbf{I}, \tag{39}$$

where $I_e = \operatorname{tr}\mathbf{e} = e$, $II_e = \frac{1}{2}\left((\operatorname{tr}\mathbf{e})^2 - \operatorname{tr}\left(\mathbf{e}^2\right)\right) = K$. This equation gives

$$\mathbf{e}^2 = \mathbf{K} + e\mathbf{e} - K\mathbf{I}. \tag{40}$$

Hence, equation (38) can be finally rewritten as

$$III_C\mathbf{b}^{-1} = \mathbf{K} - \mathbf{e} - \boldsymbol{\alpha} + (1 + e + \alpha)\mathbf{I} + O\left(\varepsilon^3\right). \tag{41}$$

Next, following the connections between the Murnaghan invariants and the principal invariants in equation (17), we have

$$\begin{aligned} J_1 &= I_C - 3 = e + \alpha, \\ J_2 &= 3 - 2I_C + II_C = K + O\left(\varepsilon^3\right), \\ J_3 &= I_C - II_C + III_C - 1 = O\left(\varepsilon^3\right). \end{aligned} \tag{42}$$

In addition, using the chain rule, we can replace the derivatives of the strain-energy function W with respect to I_C, II_C, and III_C by

$$\begin{aligned} \frac{\partial W_M}{\partial I_C} &= \frac{\partial W_M}{\partial J_1} - 2\frac{\partial W_M}{\partial J_2} + \frac{\partial W_M}{\partial J_3}, \\ \frac{\partial W_M}{\partial II_C} &= \frac{\partial W_M}{\partial J_2} - \frac{\partial W_M}{\partial J_3}, \; \frac{\partial W_M}{\partial III_C} = \frac{\partial W_M}{\partial J_3}. \end{aligned} \tag{43}$$

With respect to the third-order Murnaghan strain-energy function in equation (24), we have

$$\begin{aligned}\frac{\partial W_M}{\partial I_C} &= (a_5 - 2a_1) + 2(a_2 - a_3)J_1 + a_3 J_2 + 3a_4 J_1^2 + O\left(\varepsilon^3\right), \\ \frac{\partial W_M}{\partial II_C} &= (a_1 - a_5) + a_3 J_1 + O\left(\varepsilon^2\right), \ \frac{\partial W_M}{\partial III_C} = a_5 + O(\varepsilon).\end{aligned} \tag{44}$$

Moreover, recalling that $\mathcal{J} = \det(\mathbf{F}) = III_C^{1/2}$ and using equation (37), we have

$$\frac{2}{\mathcal{J}} = \frac{2}{\det(\mathbf{F})} = 2 - e + \frac{3}{4}e^2 - K - \alpha + O\left(\varepsilon^3\right). \tag{45}$$

Finally, substituting equations (31), (34), (37), (41), (44), and (45) into equation (23) and ignoring the higher-order terms, we can derive the second-order expansion of the Cauchy stress tensor

$$\begin{aligned}\mathbf{t} = &[-2a_1\mathbf{e} + (2a_1 + 4a_2)e\mathbf{I}] + [(a_1 + 4a_2 - 2a_3)e\mathbf{e} - (2a_1 - 2a_5)\mathbf{K} - 2a_1\boldsymbol{\alpha} \\ &+ \left((2a_1 + 2a_3)K + (2a_1 + 4a_2)\alpha - (a_1 + 2a_2 - 2a_3 - 6a_4)e^2\right)\mathbf{I}\big] + O\left(\varepsilon^3\right).\end{aligned} \tag{46}$$

From this, we extract the first-order linear elastic term

$$\mathbf{t}_1 = -2a_1\mathbf{e} + (2a_1 + 4a_2)e\mathbf{I}, \tag{47}$$

and the second-order term

$$\begin{aligned}\mathbf{t}_2 = &(a_1 + 4a_2 - 2a_3)e\mathbf{e} - (2a_1 - 2a_5)\mathbf{K} - 2a_1\boldsymbol{\alpha} \\ &+ \left((2a_1 + 2a_3)K + (2a_1 + 4a_2)\alpha - (a_1 + 2a_2 - 2a_3 - 6a_4)e^2\right)\mathbf{I}.\end{aligned} \tag{48}$$

3.2. Third-order elasticity

3.2.1. Strain-energy functions. The third-order elasticity requires expanding the stress and strain tensors up to third-order smallness $O\left(\varepsilon^3\right)$, as well as the strain-energy function up to fourth-order smallness $O\left(\varepsilon^4\right)$. The fourth-order energy function can be expressed using the Murnaghan invariants by

$$W_M = W_{2M} + W_{3M} + W_{4M} + O\left(\varepsilon^5\right), \tag{49}$$

where $W_{4M} = a_6 J_1 J_3 + a_7 J_1^2 J_2 + a_8 J_2^2 + a_9 J_1^4$ and $a_6, \ a_7, \ a_8$, and a_9 are $O(\varepsilon)$ material constants.
In addition, the fourth-order energy function can be expressed in terms of the Landau invariants as

$$W_L = W_{2L} + W_{3L} + W_{4L} + O\left(\varepsilon^5\right), \tag{50}$$

where $W_{4L} = \bar{E}\bar{I}_1\bar{I}_3 + \bar{F}\bar{I}_1^2\bar{I}_2 + \bar{G}\bar{I}_2^2 + \bar{H}\bar{I}_1^4 \ \mu$ and $\bar{E}, \ \bar{F}, \ \bar{G}$, and $\bar{H}$ are $O(\varepsilon)$ material constants.
Referring to the connections between the Murnaghan invariants and Landau invariants in equation (20), we can rewrite the fourth-order strain-energy term in the Murnaghan expansion as

$$W_{4M} = \frac{16a_6}{3}\bar{I}_1\bar{I}_3 - 8(a_6 + a_7 + a_8)\bar{I}_1^2\bar{I}_2 + 4a_8\bar{I}_2^2 + \left(\frac{8a_6}{3} + 8a_7 + 4a_8 + 16a_9\right)\bar{I}_1^4. \tag{51}$$

Thus, the relationships between the material constants are

$$\bar{E} = \frac{16a_6}{3}, \ \bar{F} = -8(a_6 + a_7 + a_8), \ \bar{G} = 4a_8, \ \bar{H} = \frac{8a_6}{3} + 8a_7 + 4a_8 + 16a_9. \tag{52}$$

Similarly, referring to the connections between the Landau invariants and Murnaghan invariants in equation (19), we can rewrite the fourth-order strain-energy term in the Landau expansion as follows:

$$W_{4L} = \frac{3\bar{E}}{16}J_1J_3 - \frac{1}{16}(3\bar{E} + 2\bar{F} + 4\bar{G})J_1^2J_2 + \frac{\bar{G}}{4}J_2^2 + \frac{1}{16}(\bar{E} + \bar{F} + \bar{G} + \bar{H})J_1^4, \tag{53}$$

which gives the relationship among the material constants

$$a_6 = \frac{3\bar{E}}{16}, \ a_7 = -\frac{1}{16}(3\bar{E} + 2\bar{F} + 4\bar{G}), \ a_8 = \frac{\bar{G}}{4}, \ a_9 = \frac{1}{16}(\bar{E} + \bar{F} + \bar{G} + \bar{H}). \tag{54}$$

3.2.2. Stress–strain relationships. Here, our aim is to expand all terms in equation (23) up to the third-order terms and obtain the third-order expansion of the Cauchy stress tensor, represented by $\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3 + O\left(\varepsilon^4\right)$, where $\mathbf{t}_3$ denotes the third-order term. By expanding the square of the left Cauchy–Green strain tensor $\mathbf{b}$, we have

$$\mathbf{b}^2 = \mathbf{I} + 2\mathbf{e} + 2\boldsymbol{\alpha} + \mathbf{e}^2 + 2\boldsymbol{\beta} + O\left(\varepsilon^4\right), \tag{55}$$

and its trace

$$\mathrm{tr}\left(\mathbf{b}^2\right) = 3 + 2e + 2\alpha + e^2 - 2K + 2\beta + O\left(\varepsilon^4\right), \tag{56}$$

where $\boldsymbol{\beta} = \mathbf{e}\boldsymbol{\alpha}$ and $\beta = \mathrm{tr}(\mathbf{e}\boldsymbol{\alpha})$. Therefore, the second principal invariant II_b can be obtained by

$$II_b = \frac{1}{2}\left((\mathrm{tr}\mathbf{b})^2 - \mathrm{tr}\left(\mathbf{b}^2\right)\right) = 3 + 2e + 2\alpha + K + \alpha e - \beta + O\left(\varepsilon^4\right). \tag{57}$$

Next, referring to equations (30) and (35) and expand to the third-order terms, we find

$$j_1 = e + \alpha,\ j_2 = K + e\alpha - \beta + O\left(\varepsilon^4\right),\ j_3 = L + O\left(\varepsilon^4\right), \tag{58}$$

where $L = \det\mathbf{e}$. Therefore, we can yield

$$III_b = 1 + e + \alpha + K + e\alpha - \beta + L + O\left(\varepsilon^4\right). \tag{59}$$

Moreover, according to the Cayley–Hamilton theorem, we have

$$\begin{aligned} III_C\mathbf{b}^{-1} = \mathbf{K} - \mathbf{e} - \boldsymbol{\alpha} + (1 + e + \alpha + \alpha e - \beta)\mathbf{I} \\ + 2\boldsymbol{\beta} - e\boldsymbol{\alpha} - \alpha\mathbf{e} + O\left(\varepsilon^4\right). \end{aligned} \tag{60}$$

By following the connections between the Murnaghan invariants and principal invariants in equation (17) and expanding to third-order terms, we find

$$\begin{aligned} J_1 &= I_C - 3 = e + \alpha \\ J_2 &= 3 - 2I_C + II_C = K + \alpha e - \beta + O\left(\varepsilon^4\right) \\ J_3 &= I_C - II_C + III_C - 1 = L + O\left(\varepsilon^4\right) \end{aligned} \tag{61}$$

In addition, using the chain rule, we can replace the derivatives of the strain-energy function W_M with respect to I_C, II_C, and III_C by

$$\begin{aligned} \frac{\partial W_M}{\partial I_C} &= \frac{\partial W_M}{\partial J_1} - 2\frac{\partial W_M}{\partial J_2} + \frac{\partial W_M}{\partial J_3}, \\ \frac{\partial W_M}{\partial II_C} &= \frac{\partial W_M}{\partial J_2} - \frac{\partial W_M}{\partial J_3},\ \frac{\partial W_M}{\partial III_C} = \frac{\partial W_M}{\partial J_3}. \end{aligned} \tag{62}$$

In the case of the fourth-order Murnaghan strain-energy function in equation (49), we have

$$\begin{aligned} \frac{\partial W_M}{\partial I_C} &= (a_5 - 2a_1) + (2a_2 - 2a_3 + a_6)J_1 + (a_3 - 4a_8)J_2 \\ &\quad +(3a_4 - 2a_7)J_1^2 + 4a_9J_1^3 + 2a_7J_1J_2 + a_6J_3 + O\left(\varepsilon^4\right), \\ \frac{\partial W_M}{\partial II_C} &= (a_1 - a_5) + (a_3 - a_6)J_1 + a_7J_1^2 + 2a_8J_2 + O\left(\varepsilon^3\right), \\ \frac{\partial W_M}{\partial III_C} &= a_5 + a_6J_1 + O\left(\varepsilon^2\right). \end{aligned} \tag{63}$$

Moreover, recalling that $\mathcal{J} = \det(\mathbf{F}) = III_C^{1/2}$ and using equation (59), we have

$$\frac{2}{\mathcal{J}} = 2 - e + \frac{3}{4}e^2 - K - \alpha - \frac{5}{8}e^3 + \frac{3}{2}eK + \frac{1}{2}e\alpha + \beta - L + O\left(\varepsilon^4\right). \tag{64}$$

Finally, substituting the equations (31), (57), (59), (60), (64), and (64) into equation (23) and ignoring the higher-order terms, we can obtain the Cauchy stress up to the third-order terms to be

$$\mathbf{t} = \mathbf{t}_1 + \mathbf{t}_2 + \mathbf{t}_3 + O\left(\varepsilon^4\right) \tag{65}$$

in which the third-order term is

$$\begin{aligned}\mathbf{t}_3 = &\left[\left(-\frac{3}{4}a_1 - 2a_2 + a_3 + 6a_4 - 2a_7\right)e^2 + (a_1 + 2a_3 - 4a_8)K\right.\\ &\left.+ (3a_1 + 4a_2 - 2a_3 - 2a_5)\alpha\right]\mathbf{e} + (a_1 - 2a_3 - a_5 + 2a_6)e\mathbf{K}\\ &+ (3a_1 + 4a_2 - 2a_3 - 2a_5)e\boldsymbol{\alpha} + (4a_5 - 4a_1)\boldsymbol{\beta}\\ &+ \left(\left(\frac{3}{4}a_1 + \frac{3}{2}a_2 - a_3 - 3a_4 + 2a_7 + 8a_9\right)e^3 + (-2a_1 - 2a_2 + a_3 + 4a_7 + 4a_8)eK\right.\\ &\left.+(-2a_1 - 4a_2 + 6a_3 + 12a_4 + 2a_5)e\alpha + (2a_5 + 2a_6)L - (2a_3 + 2a_5)\beta\right)\mathbf{I}.\end{aligned} \tag{66}$$

4. Weekly nonlinear elasticity for isotropic incompressible materials

Most soft biological materials are assumed to be incompressible, so that $III_C = 1$, and then the strain-energy function W is just I_C and II_C and the Cauchy stress tensor is given by

$$\mathbf{t} = -p\mathbf{I} + 2\frac{\partial W}{\partial I_C}\mathbf{b} - 2\frac{\partial W}{\partial II_C}\mathbf{b}^{-1}, \tag{67}$$

where p is the Lagrange multiplier that needs to be determined by a boundary condition. Given $III_C = 1$, then

$$\bar{I}_1 = -\bar{I}_1^2 + \bar{I}_2 + 2\bar{I}_1\bar{I}_2 - \frac{2}{3}\bar{I}_1^3 - \frac{4}{3}\bar{I}_3, \text{ and } J_1 = -J_2 - J_3. \tag{68}$$

In this section, we present the weakly nonlinear expansion of energy functions for commonly used incompressible isotropic hyperelastic solids in the Landau and Murnaghan forms. These include the neo-Hookean model W_{NH}, the two-parameter Mooney–Rivlin model W_{MR2}, and the five-parameter Mooney–Rivlin model W_{MR5}, which are given by

$$\begin{aligned}W_{\mathrm{NH}} &= C_{10}(I_C - 3),\\ W_{\mathrm{MR2}} &= C_{10}(I_C - 3) + C_{01}(II_C - 3),\\ W_{\mathrm{MR5}} &= C_{10}(I_C - 3) + C_{01}(II_C - 3) + C_{20}(I_C - 3)^2\\ &\quad + C_{11}(I_C - 3)(II_C - 3) + C_{02}(II_C - 3)^2.\end{aligned} \tag{69}$$

4.1. Second-order elasticity

4.1.1. Strain-energy functions. For the incompressible materials, the equation (68) indicates that $\bar{I}_1 = \frac{1}{2}(e + \alpha)$ and $J_1 = e + \alpha$ are second-order $O\left(\varepsilon^2\right)$ quantities. In addition, recalling that the strain-energy function should be expanded up to the third-order smallness for the second-order elasticity theory, the incompressibility condition in terms of the Landau invariants can be written as

$$\bar{I}_1 = \bar{I}_2 - \frac{4}{3}\bar{I}_3 + O\left(\varepsilon^4\right). \tag{70}$$

Thus, we have

$$\begin{aligned}I_C - 3 &= 2\bar{I}_1 = 2\bar{I}_2 - \frac{8}{3}\bar{I}_3 + O\left(\varepsilon^4\right),\\ II_C - 3 &= 4\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2 = 2\bar{I}_2 - \frac{16}{3}\bar{I}_3 + O\left(\varepsilon^4\right).\end{aligned} \tag{71}$$

Substituting this equation in equation (69), we can obtain the expansion form of the energy functions as follows:

$$\begin{aligned} W_{\mathrm{NH}} &= 2C_{10}\bar{I}_2 - \frac{8}{3}C_{10}\bar{I}_3 + O\left(\varepsilon^4\right), \\ W_{\mathrm{MR2}} &= 2(C_{10} + C_{01})\bar{I}_2 - \frac{8}{3}(C_{10} + 2C_{01})\bar{I}_3 + O\left(\varepsilon^4\right), \\ W_{\mathrm{MR5}} &= 2(C_{10} + C_{01})\bar{I}_2 - \frac{8}{3}(C_{10} + 2C_{01})\bar{I}_3 + O\left(\varepsilon^4\right). \end{aligned} \tag{72}$$

Following the third-order incompressible isotropic elasticity analysis by Destrade and Ogden [19], the weakly nonlinear expansion of the strain-energy functions in terms of Landau invariants up to third-order is

$$W_{\mathrm{L}} = \mu\bar{I}_2 + \frac{\bar{A}}{3}\bar{I}_3 + O\left(\varepsilon^4\right), \tag{73}$$

From this, we can obtain the following connections between the material constants of the incompressible neo-Hookean, two-parameter Mooney–Rivlin and five-parameter Mooney–Rivlin solids

$$\begin{aligned} W_{\mathrm{NH}}: &\quad \mu = 2C_{10}, \quad \bar{A} = -8C_{10}, \\ W_{\mathrm{MR2}}: &\quad \mu = 2(C_{10} + C_{01}), \quad \bar{A} = -8(C_{10} + 2C_{01}), \\ W_{\mathrm{MR5}}: &\quad \mu = 2(C_{10} + C_{01}), \quad \bar{A} = -8(C_{10} + 2C_{01}). \end{aligned} \tag{74}$$

Next, recalling the connections between I_C, II_C, III_C and J_1, J_2, J_3 in equation (14), the incompressibility condition in equation (68) can be expressed in terms of the Murnaghan invariants as follows:

$$\begin{aligned} I_C - 3 &= J_1 = -J_2 - J_3, \\ II_C - 3 &= 2J_1 + J_2 = -J_2 - 2J_3. \end{aligned} \tag{75}$$

Substituting equation (75) into equation (69) and neglecting the terms of $O\left(\varepsilon^4\right)$, we can rewrite the strain-energy functions as

$$\begin{aligned} W_{\mathrm{NH}} &= -C_{10}J_2 - C_{10}J_3, \\ W_{\mathrm{MR2}} &= -(C_{10} + C_{01})J_2 - (C_{10} + 2C_{01})J_3, \\ W_{\mathrm{MR5}} &= -(C_{10} + C_{01})J_2 - (C_{10} + 2C_{01})J_3 + O\left(\varepsilon^4\right). \end{aligned} \tag{76}$$

According to equations (73) and (29), the strain-energy functions in terms of Murnaghan invariants becomes

$$W_{\mathrm{M}} = a_1J_2 + a_5J_3 + O\left(\varepsilon^4\right). \tag{77}$$

Thus, for the incompressible neo-Hookean, two-parameter Mooney–Rivlin and five-parameter Mooney–Rivlin models, we have the following connections between the material constants:

$$\begin{aligned} W_{\mathrm{NH}}: &\quad a_1 = -C_{10}, \quad a_5 = -C_{10}, \\ W_{\mathrm{MR2}}: &\quad a_1 = -(C_{10} + C_{01}), \quad a_5 = -(C_{10} + 2C_{01}), \\ W_{\mathrm{MR5}}: &\quad a_1 = -(C_{10} + C_{01}), \quad a_5 = -(C_{10} + 2C_{01}). \end{aligned} \tag{78}$$

4.1.2. Stress–strain relationships. To simplify the calculation, we use the third-order Murnaghan form strain-energy function to derive the second-order Cauchy stress tensor. First, using the chain rule, we can rewrite the derivatives of the strain-energy function W with respect to I_C, II_C, and III_C as

$$\begin{aligned} \frac{\partial W_{\mathrm{M}}}{\partial I_C} &= -2\frac{\partial W_{\mathrm{M}}}{\partial J_2} + \frac{\partial W_{\mathrm{M}}}{\partial J_3} = -2a_1 + a_5, \\ \frac{\partial W_{\mathrm{M}}}{\partial II_C} &= \frac{\partial W_{\mathrm{M}}}{\partial J_2} - \frac{\partial W_{\mathrm{M}}}{\partial J_3} = a_1 - a_5. \end{aligned} \tag{79}$$

According to equation (37), the incompressibility condition $III_C = 1$ gives

$$e = -\alpha - K + O\left(\varepsilon^3\right), \tag{80}$$

which indicates that e is the second-order quantity. Hence, equation (41) can be reduced to

$$\mathbf{b}^{-1} = \mathbf{K} - \mathbf{e} - \boldsymbol{\alpha} + (1-K)\mathbf{I} \ + O\left(\varepsilon^3\right). \tag{81}$$

Therefore, substituting the equations (79) and (81) into equation (67) and ignoring the higher-order terms, we have the second-order Cauchy stress tensor

$$\mathbf{t} = -2a_1\mathbf{e} - (2a_1 - 2a_5)\mathbf{K} - 2a_1\boldsymbol{\alpha} + (-6a_1 + 4a_5 + (2a_1 - 2a_5)K - p)\mathbf{I} + O\left(\varepsilon^3\right). \tag{82}$$

Specifically, according to the equation (78), for incompressible neo-Hookean solid, the Cauchy stress tensor is given by

$$\mathbf{t} = 2\mathrm{C}_{10}\mathbf{e} + 2\mathrm{C}_{10}\boldsymbol{\alpha} + (2C_{10} - p)\mathbf{I} + O\left(\varepsilon^3\right). \tag{83}$$

For incompressible two-parameter Mooney–Rivlin solid, the Cauchy stress tensor can be rewritten as

$$\mathbf{t} = 2(\mathrm{C}_{10} + \mathrm{C}_{01})\mathbf{e} + 2(\mathrm{C}_{10} + \mathrm{C}_{01})\boldsymbol{\alpha} - 2\mathrm{C}_{01}\mathbf{K} + (2C_{10} - 2\mathrm{C}_{01}(1-K) - p)\mathbf{I} + O\left(\varepsilon^3\right), \tag{84}$$

and for incompressible five-parameter Mooney–Rivlin solid, the Cauchy stress tensor can be expressed by

$$\mathbf{t} = 2(\mathrm{C}_{10} + \mathrm{C}_{01})\mathbf{e} + 2(\mathrm{C}_{10} + \mathrm{C}_{01})\boldsymbol{\alpha} - 2\mathrm{C}_{01}\mathbf{K} + (2C_{10} - 2\mathrm{C}_{01}(1-K) - p)\mathbf{I} + O\left(\varepsilon^3\right). \tag{85}$$

4.2. Third-order elasticity

4.2.1. Strain-energy functions. For the third-order elasticity, the strain-energy function is expanded to the fourth order. The incompressibility condition, in terms of the Landau invariants, is

$$\bar{I}_1 = \bar{I}_2 - \frac{4}{3}\bar{I}_3 + \bar{I}_2^2 + O\left(\varepsilon^4\right). \tag{86}$$

Therefore,

$$\begin{aligned} I_C - 3 &= 2\bar{I}_1 = 2\bar{I}_2 - \frac{8}{3}\bar{I}_3 + 2\bar{I}_2^2 + O\left(\varepsilon^4\right) \\ II_C - 3 &= 4\bar{I}_1 + 2\bar{I}_1^2 - 2\bar{I}_2 = 2\bar{I}_2 - \frac{16}{3}\bar{I}_3 + 6\bar{I}_2^2 + O\left(\varepsilon^4\right). \end{aligned} \tag{87}$$

Hence, we can rewrite equation (69) as

$$\begin{aligned} W_{\mathrm{NH}} &= 2C_{10}\bar{I}_2 - \frac{8}{3}C_{10}\bar{I}_3 + 2C_{10}\bar{I}_2^2 + O\left(\varepsilon^4\right) \\ W_{\mathrm{MR2}} &= 2(C_{10} + C_{01})\bar{I}_2 - \frac{8}{3}(C_{10} + 2C_{01})\bar{I}_3 + 2(3C_{01} + C_{10})\bar{I}_2^2 + O\left(\varepsilon^4\right) \\ W_{\mathrm{MR5}} &= 2(C_{10} + C_{01})\bar{I}_2 - \frac{8}{3}(C_{10} + 2C_{01})\bar{I}_3 \\ &\quad + 2(3C_{01} + C_{10} + 2C_{20} + 2C_{11} + 2C_{02})\bar{I}_2^2 + O\left(\varepsilon^4\right). \end{aligned} \tag{88}$$

Following Destrade and Ogden [19], the weakly nonlinear expansion of the strain-energy functions in terms of Landau invariants to fourth order is

$$W_{\mathrm{L}} = \mu\bar{I}_2 + \frac{\bar{A}}{3}\bar{I}_3 + \bar{D}\bar{I}_2^2 + O\left(\varepsilon^5\right), \tag{89}$$

where $\bar{D} = \lambda/2 + \bar{B} + \bar{G}$. Hence, the corresponding material constants of the incompressible neo-Hookean, two-parameter Mooney–Rivlin and five-parameter Mooney–Rivlin solids are as follows:

$$\begin{aligned} W_{\mathrm{NH}} &: \mu = 2C_{10},\ \bar{A} = -8C_{10},\ \bar{D} = 2C_{10} \\ W_{\mathrm{MR2}} &: \mu = 2(C_{10} + C_{01}),\ \bar{A} = -8(C_{10} + 2C_{01}),\ \ \bar{D} = 2(3C_{01} + C_{10}) \\ W_{\mathrm{MR5}} &: \mu = 2(C_{10} + C_{01}),\ \bar{A} = -8(C_{10} + 2C_{01}), \\ &\quad \bar{D} = 2(3C_{01} + C_{10} + 2C_{20} + 2C_{11} + 2C_{02}). \end{aligned} \tag{90}$$

To represent the strain-energy function in terms of the Murnaghan expansion, we substitute equation (75) into equation (69) to get

$$\begin{aligned} W_{\mathrm{NH}} &= -C_{10}J_2 - C_{10}J_3, \\ W_{\mathrm{MR2}} &= -(C_{10} + C_{01})J_2 - (C_{10} + 2C_{01})J_3, \\ W_{\mathrm{MR5}} &= -(C_{10} + C_{01})J_2 - (C_{10} + 2C_{01})J_3 + (C_{20} + C_{11} + C_{02})J_2^2 + O\left(\varepsilon^4\right). \end{aligned} \tag{91}$$

Similarly, from equations (89) and (29), the expansion of the strain-energy functions in terms of Murnaghan invariants is

$$W_{\mathrm{M}} = a_1J_2 + a_5J_3 + b_1J_2^2 + O\left(\varepsilon^4\right), \tag{92}$$

where b_1 is $O(\varepsilon)$ material constant. Therefore, for the incompressible neo-Hookean, two-parameter Mooney–Rivlin and five-parameter Mooney–Rivlin solids, we have the relationships between the material constants as

$$\begin{aligned} W_{\mathrm{NH}} &: a_1 = -C_{10},\ a_5 = -C_{10},\ b_1 = 0, \\ W_{\mathrm{MR2}} &: a_1 = -(C_{10} + C_{01}),\ a_5 = -(C_{10} + 2C_{01}),\ b_1 = 0, \\ W_{\mathrm{MR5}} &: a_1 = -(C_{10} + C_{01}),\ a_5 = -(C_{10} + 2C_{01}),\ b_1 = (C_{20} + C_{11} + C_{02}). \end{aligned} \tag{93}$$

4.2.2. Stress-strain relationships in third-order elasticity. Similarly, we use the fourth-order strain-energy function in Murnaghan expansion to solve the third-order Cauchy stress tensor. Then, based on equation (92), we have

$$\begin{aligned} \frac{\partial W_{\mathrm{M}}}{\partial I_C} &= -2\frac{\partial W_{\mathrm{M}}}{\partial J_2} + \frac{\partial W_{\mathrm{M}}}{\partial J_3} = -2a_1 + a_5 - 4b_1J_2, \\ \frac{\partial W_{\mathrm{M}}}{\partial II_C} &= \frac{\partial W_{\mathrm{M}}}{\partial J_2} - \frac{\partial W_{\mathrm{M}}}{\partial J_3} = a_1 - a_5 + 2b_1J_2. \end{aligned} \tag{94}$$

Referring to equation (59), the incompressibility condition $III_C = 1$ gives

$$e = -\alpha - K + \beta - L + O\left(\varepsilon^4\right) \tag{95}$$

With this equation, equation (60) can be rewritten as

$$\mathbf{b}^{-1} = \mathbf{K} - \mathbf{e} - \boldsymbol{\alpha} + (1 - K - L)\mathbf{I} - 2\boldsymbol{\beta} + \alpha\mathbf{e} + O\left(\varepsilon^4\right). \tag{96}$$

In addition, considering the e is $O\left(\varepsilon^2\right)$, the J_2 in equation (61) can be reduced to

$$J_2 = K - \beta + O\left(\varepsilon^4\right). \tag{97}$$

Therefore, substituting equations (94) and (96) into equation (67), we can derive the Cauchy stress tensor by

$$\begin{aligned} \mathbf{t} = &-2a_1\mathbf{e} - (2a_1 - 2a_5)\mathbf{K} - 2a_1\boldsymbol{\alpha} - 4b_1K\mathbf{e} + (2\mathrm{a}_1 - 2\mathrm{a}_5)\alpha\mathbf{e} - (4\mathrm{a}_1 - 4\mathrm{a}_5)\boldsymbol{\beta} \\ &+ (-6a_1 + 4a_5 + (2a_1 - 2a_5 - 12b_1)K + (2a_1 - 2a_5)L - 12b_1\beta - p)\mathbf{I}. \end{aligned} \tag{98}$$

Specifically, according to equation (93), for incompressible neo-Hookean solid, the Cauchy stress tensor is given by

$$\mathbf{t} = 2\mathrm{C}_{10}\mathbf{e} + 2\mathrm{C}_{10}\boldsymbol{\alpha} + (2C_{10} - p)\mathbf{I} + O\left(\varepsilon^4\right). \tag{99}$$

For incompressible two-parameter Mooney–Rivlin solid, the Cauchy stress tensor is given by

$$\begin{aligned}\mathbf{t} = {} & 2(\mathrm{C}_{10} + \mathrm{C}_{01})\mathbf{e} + 2(\mathrm{C}_{10} + \mathrm{C}_{01})\boldsymbol{\alpha} - 2\mathrm{C}_{01}\mathbf{K} + 2C_{01}\alpha\mathbf{e} - 4C_{01}\boldsymbol{\beta} \\ & + (2C_{10} - 2\mathrm{C}_{01}(1 - K - L) - p)\mathbf{I} + O\left(\varepsilon^4\right).\end{aligned} \tag{100}$$

For incompressible five-parameter Mooney–Rivlin solid, the Cauchy stress tensor is given by

$$\begin{aligned}\mathbf{t} = {} & 2(\mathrm{C}_{10} + \mathrm{C}_{01})\mathbf{e} + 2(\mathrm{C}_{10} + \mathrm{C}_{01})\boldsymbol{\alpha} - 2\mathrm{C}_{01}\mathbf{K} \\ & - 4(C_{20} + C_{11} + C_{02})K\mathbf{e} + 2C_{01}\alpha\mathbf{e} - 4C_{01}\boldsymbol{\beta} \\ & + (2C_{10} - 2\mathrm{C}_{01}(1 - K - L) + 12(C_{20} + C_{11} + C_{02})(\beta - K) - p)\mathbf{I} + O\left(\varepsilon^4\right).\end{aligned} \tag{101}$$

5. Conclusion

The nonlinear elastic behavior of soft materials is of significant importance across many fields, including biology, materials science, geophysics, and acoustics. In this paper, we have given new results for the expansion of the strain-energy functions and Cauchy stress tensor to $O\left(\varepsilon^4\right)$ where $\varepsilon = \sqrt{\mathbf{H} \cdot \mathbf{H}}$ and $0 < \varepsilon \leq 1$ for the weakly nonlinear asymptotic expansion for small perturbations to the deformation gradient tensor $\mathbf{F} = \mathbf{I} + \mathbf{H}$. These theories provide us with powerful tools for understanding and predicting the mechanical response of soft materials under complex loading conditions.

By examining distinct invariants of strain tensors, strain-energy functions, stress–strain relations, and transformation relations of material parameters, we reveal the connections between different elastic theories and expand the energy density function to third-order and fourth-order terms under the framework of weak nonlinear theory. Such efforts not only guide further research on the elastic behavior of soft materials but also contribute to finding solutions for practical problems. It is worth highlighting that this paper also addresses the strain-energy function and stress–strain relationship of soft materials under incompressible conditions. This consideration facilitates the modeling and analysis of practical problems while providing a simplified approach to tackling complex problems.

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